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A HYPERBOLIC SYSTEM FROM A NEURAL TRANSMISSION MODEL

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Abstract—We provide a global existence result for a hyperbolic system which arises in the theory of neural conduction.

We shall be concerned with the following system which couples a hyperbolic equation with first-order equations:

$$\partial^2 V/\partial t^2 - \partial^2 V/\partial x^2 = L_1(W_1, W_2, W_3)\partial V/\partial t + L_2(W_1, W_2, W_3)V + L_3(W_1, W_2, W_3), \tag{1a}$$

$$\partial W_i/\partial t = \alpha_i(V) - \beta_i(V)W_i, \tag{1b}$$

for $i = 1, 2, 3,\ x \in R$ and $t \geqslant 0$; with initial conditions of the form

$$V(x, 0) = V_1(x), \tag{1c}$$

$$V_t(x, 0) = V_2(x), \tag{1d}$$

$$W_i(x, 0) = W_i(x). \tag{1e}$$

We assume that the functions $L_i(\ldots)$: $R^3 \to R^3$ are continuously differentiable in each place and that $\alpha_i(\ )$, $\beta_i(\ )$: $R \to R$ are uniformly bounded, positive and continuously differentiable. We obtain results which guarantee the global existence of solutions to equations (1a–e).

The source of our motivation for studying equations (1a–e) is the neural conduction model appearing in Ref. [1]. Basically, it is a Hodgkin–Huxley model which incorporates a nontrivial induction term. Hodgkin–Huxley models having zero induction have been the subject of thorough mathematical study and the interested reader is referred to Refs [2–6].

It is convenient to rewrite equations (1a–e) as a first-order system. If we set $u = V_x$ and $v = V_t$, we obtain the following equivalent system:

$$\partial u/\partial t - \partial v/\partial x = 0, \tag{2a}$$

$$\partial v/\partial t - \partial u/\partial x = L_1(W_1, W_2, W_3)v + L_2(W_1, W_2, W_3)V + L_3(W_1, W_2, W_3), \tag{2b}$$

$$\partial W_i/\partial t = \alpha_i(V) - \beta_i(V)W_i; \tag{2c}$$

with initial conditions

$$u(x, 0) = u_0(x) = V_1'(x), \tag{2d}$$

$$v(x, 0) = v_0(x) = V_2(x), \tag{2e}$$

$$W_i(x, 0) = W_i(x). \tag{2f}$$

In order to write equations (2a–f) as a Banach-space-valued ordinary differential equation, we need the following regularity lemma.

Lemma 1

Suppose that $X = X_1 \oplus X_2$ is a Cartesian product of Banach spaces and that A generates a C^0-group, $\{T(t) \mid t \in R\}$ on X. Let the scalar operator $f(t)$ be defined by the diagonal matrix having $f_1(t)$ and $f_2(t)$ along the diagonal and let $g(t) = \langle g_1(t), g_2(t)\rangle^T$. Assume that $f_i(\ )$: $[0, T] \to R$ and

$g_i(\ ): [0, T] \to X_i$ are continuously differentiable. If $x_0 = (u_0, v_0) \in D(A)$ then there exists a unique $z(\ ): [0, T] \to X$, such that $z(0) = x_0$ and

$$z(t) = T(t)x_0 + \int_0^t T(t-s)[f(s)u(s) + g(s)]\,\mathrm{d}s. \tag{3}$$

Moreover, $z(\ )$ is continuously differentiable on $[0, T]$ and

$$\dot{z}(t) = Az(t) + f(t)z(t) + g(t). \tag{4}$$

Proof. We let $F(t)$ be the diagonal matrix having

$$\exp\left[\int_0^t f_i(s)\,\mathrm{d}s\right]$$

along the diagonal and $u(\ )$ denote the unique continuously differentiable solution to $\dot{u}(t) = Au(t) + [F(t)^{-1}g(t)]$ [cf. 7, p. 486], setting $z(t) = F(t)u(t)$ we obtain our desired solution.

We let X_0 denote the space of bounded uniformly continuous real-valued functions endowed with the norm $\|u\| = \sup|u(x)|$. We define A_0; $X_0 \to X_0$ by $(A_0 u)(x) = u'(x)$ for $u \in D(A_0) = \{u \in X_0 | u' \in X_0\}$. Letting $X_1 = X_0 \oplus X_0$ with the L_1 sum norm, we define A_1: $X_1 \to X_1$ as the matrix operator:

$$A_1 = \begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix} \quad \text{on} \quad D(A_1) = D(A_0) \oplus D(A_0). \tag{5}$$

If $u, v, W_i \in X_0$, we make the obvious identifications $(u)(x) = u(x)$, $(v)(x) = v(x)$ and $(W_i)(x) = W_i(x)$. We have the following theorem.

Theorem 1

Let $V_1, V_2 \in X_0$ and $V_i(x) = u_0(x)$ and $V_2(x) = V_0(x)$ for $x \in R$. If $T > 0$ and $(u_0, v_0) \in D(A_1)$ and $W_{i0} \in X_0$, then there exists a unique $u(\ ), v(\ ), W_i(\ )$ $(i = 1, 2, 3)$ mapping $[0, T]$ to X_0 satisfying equations (2a–f). Furthermore, $(u(t), v(t)) \in D(A_1)$ for $t \in [0, T]$.

Indication of proof. We first adopt arguments appearing in Ref. [8] to show that A_1 is the infinitesimal generator of a strongly continuous group $\{T_1(t) | t \in R\}$ of class $G(2, 0, X_0)$. realizing that

$$V = V_1 + \int_0^t v(s)\,\mathrm{d}s,$$

we see that equations (2a–f) become a system of integrodifferential equations. Solutions to this system can be shown to satisfy the following Volterra system:

$$u(t) = \pi_1 T_1(t)u_0, \tag{6a}$$

$$\begin{aligned} v(t) = \pi_2 T_1(t)v_0 + \int_0^t \pi_2 T_1(t-s)\Big\{&L_1(W_1(s)W_2(s)W_3(s))v(s) \\ &+ L_2(W_1(s), W_2(s), W_3(s))\left[V_1 + \int_0^s v(s)\,\mathrm{d}s\right] + L_3(W_1(s), W_2(s), W_3(s))\Big\}\,\mathrm{d}s, \end{aligned} \tag{6b}$$

$$\begin{aligned} W_i(t) = W_{i0} &\int_0^t \exp\left[\beta_i\left(V_1 + \int_0^s v(s)\,\mathrm{d}s\right)\right]\mathrm{d}s \\ &\times \int_0^t \exp\left\{\int_s^t \beta_i\left(V_1 + \int_0^r v(r)\,\mathrm{d}r\right)\left[\alpha_i\left(V + \int_0^r v(r)\,\mathrm{d}r\right)\right]\right\}\mathrm{d}s. \end{aligned} \tag{6c}$$

We obtain a local solution on an interval $[0, T_0]$ $(T_0 \leqslant T)$ to equations (6a–c) by using the contraction mapping theorem on a subset of the Banach space $C([0, T],_{i=1} \oplus^5 (X_0)_i)$.

To see that solutions to equations (6a–c) are differentiable, we first observe that the $W_i(\ )$ are continuously differentiable in t. Thus, the expressions $L_i(W_1(t), W_2(t), W_3(t))$ are continuously differentiable in t and we may observe that equations (1a, b) can be written in the form

$$\dot{z}(t) = A_1 z(t) + f(t)z(t) + g(t),$$

in the space X_1, where $f(t)$ is a scalar diagonal matrix with continuously differentiable $f_i(t)$ along the diagonal and $g(t) = \langle g_1(t), g_2(t)\rangle^T$ is continuously differentiable. Consequently, we may apply Lemma 1 to conclude that equations (6a, b) are continuously differentiable and satisfy equation (5).

To extend our solutions we assume that a solution exists on an open interval $[0, T^*)$, The boundedness of $\alpha_i(\ )$ and $\beta_i(\ )$ implies that $W_i(t)$ is uniformly bounded on $[0, T^*)$ and hence the expressions $l_i(W_1(t), W_2(t), W_3(t))$ will be uniformly bounded on $[0, T^*)$. We now utilize a Gronwall argument to show that $v(t)$ is uniformly bounded. This boundedness together with the integral representation (6a–c) allows us to compute the limits of $u(t)$, $v(t)$ and $W_i(t)$ as $t \to T^*$. Lemma 1 insures that $(u(T^*), v(T^*)) \in D(A_1)$ and our local existence theory will allow us to carry solutions beyond an arbitrary T^*. We may assume that our solutions exist on $[0, T]$ for any $T > 0$.

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